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BACK SCATTERING OF A PLANE ELECTROMAGNETIC WAVE FROM AN INFINITE CYLINDER OF PLASMA WITH SMALL AXIAL GRADIENTS AND ARBITRARY RADIAL GRADIENTS OF ELECTRON DENSITY - NORMAL INCIDENCE

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I DESCRIPTION AND FORMULATION OF PROBLEM

The problem of scattering of plane electromagnetic waves by cylindrical plasma structures without axial variation of plasma properties and with or without radial variation has been amenable to more or less classical diffraction analysis and has been treated by several investigators over the past few years [1, 2, 3, 4]. Axial uniformity of the plasma structure simplifies these analyses tremendously by rendering them essentially two-dimensional. Even under the case of obliquely incident plane waves, the axial dependence of the solution is set by the axial dependence of the incident wave, and the axial coordinate effectively disappears from the problem.

The introduction of axial property variation of the plasma adds essentially to the complexity of the problem and requires a three-dimensional treatment.

Moreover, axial dependence of plasma properties alters essentially the nature of scattered field. An obliquely incident plane wave exhibits no back scattering from an axially uniform plasma structure, whereas axial nonuniformity allows the possibility of back scatter. A normally incident plane wave exhibits purely radial scattering from an axially uniform plasma structure, whereas axial nonuniformity introduces axial as well as radial scattering (hence the essentially 3-dimensional nature of the problem).

In an effort to simulate more closely the characteristics of ionized missile wakes in their effects on incident electromagnetic radiation, one is eventually forced to introduce axial nonuniformity since the axially uniform configuration is considerably over-idealized. The present report describes an approximate diffraction calculation for cylindrical plasma structures having arbitrary (axisymmetric) radial property variation and weak axial property variation (small axial gradients of electron density).

A plane wave is assumed normally incident upon a cylindrical configuration such that the incident wave has its electric vector parallel to the cylinder axis (fig. 1a). The electron density distribution in the plasma is assumed to be axisymmetric arbitrarily distributed radially, but slowly varying axially (fig. 1b).

Under harmonic time-dependence $e^{-i\omega t}$ the field equations appropriate to the problems are usually given in the form:

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H} \tag{1}$$

$$\nabla \times \vec{H} = -i\omega\,\epsilon_o \vec{E} + \frac{n e^2 \vec{E}}{m(\nu - i\omega)} \tag{2}$$

here $\vec{E}$ and $\vec{H}$ are respectively the electric and magnetic field vectors, μ_o, ϵ_o are respectively the magnetic and dielectric permeabilities in a vacuum, n is electron number density, e is electron mass, and ν is the collision frequency for electrons with neutral particles.

Introducing plasma frequency ω_p, defined by

$$\omega_p^2 = \frac{e^2 n}{\epsilon_o m} \tag{3}$$

Eq. (2) takes the form

$$\nabla \times \vec{H} = -i\omega \epsilon_o \vec{E}\left(1-\frac{\omega_p^2}{\omega^2+\nu^2}\left(1-\frac{i\nu}{\omega}\right)\right) \tag{4}$$

The field equations (1) (4) are supplemented by the statement that the problem is forced by an incident plane wave of the form

$$\vec{E}_{inc} = \begin{pmatrix} 0 \\ 0 \\ E_o \end{pmatrix} e^{-i\omega t - ik_o x} \tag{5}$$

where

$$k_o = \frac{\omega}{c} = \omega\sqrt{\epsilon_o \mu_o} \tag{6}$$

and by the condition that a scattered field results in the vacuum assumed to exist outside the plasma structure, with continuity required between tangential field components ($\vec{H}$ and $\vec{E}$) across the plasma-vacuum interface

$$[\vec{E}_{tang}] = [\vec{H}_{tang}] = 0 \tag{7}$$

where [ ] signifies jump in a quantity across the interface. Moreover, the scattered field must satisfy the radiation or outgoing-wave condition at large distances from the cylinder.

II SOLUTION METHOD

A. Review of Axially Uniform Plasma Case

In the case of an axially uniform plasma, the scattered field, as well as the field inside the plasma, has a purely axial electric vector and, since normal incidence is assumed, no z-dependence enters the problem. Thus $\vec{E}$ is divergence-less and

$$\nabla \times \nabla \times \vec{E} = -\Delta \vec{E} = k_o^2 \vec{E}\left(1-\frac{\omega_p^2}{\omega^2+\nu^2}\left(1-\frac{i\nu}{\omega}\right)\right) \qquad (8)_u^*$$

Furthermore, since $\vec{E}$ is purely axial and, this equation (8) applies component by component in cartesian coordinates, the equation of the problem, under axial uniformity of plasma structure, reduces to

$$\Delta \vec{E} + k_o^2\left(1-P(r)\right)\vec{E} = 0 \qquad (9)_u$$

where $\vec{E}$ represents $\vec{E}_z$

$$\text{and where } P(r) = \frac{\omega_p^2}{\omega^2+\nu^2}\left(1-\frac{i\nu}{\omega}\right) \qquad (10)_u$$

In the vacuum, of course, P reduces to zero.

The magnetic field is expressible, via equation (1), by

$$\vec{H} = \frac{1}{i\omega\mu_o} \nabla \times \vec{E} \tag{1}$$

both inside and outside the plasma.

*A subscript u on an equation number indicates that the equation holds only in the case of an axially uniform plasma and not in the presence of axial nonuniformity.

Switching, for convenience, to cylindrical polar coordinates (r, θ, z), the incident wave (5) takes the form (dropping the harmonic time-dependence henceforth)

$$E_{inc.} = E_o e^{-ik_o(r\cos\theta)}$$
$$= E_o \sum_{m=-\infty}^{\infty} e^{-im\pi/2} J_m(k_o r) e^{im\theta} \qquad (11)$$

This must be (and is), of course, a solution of (9) with $P(r) \equiv 0$. The scattered electric field must then have the form

$$E_{scat.} = \sum_{m=-\infty}^{\infty} c_m H_m^{(1)}(k_o r) e^{im\theta} \qquad (12)_u$$

since this is the general solution of (9) with $P \equiv 0$ which satisfies the radiation condition at large r. Here the c_m are constant coefficients, and the axially uniform plasma problem then reduces to the problem of determining these coefficient c_m. The magnetic field, corresponding to purely axial electric field is given by

$$\vec{H} = \begin{vmatrix} H_r \\ H_\theta \\ H_z \end{vmatrix} = \frac{1}{i\omega\mu_o} \begin{pmatrix} \frac{1}{r}\frac{\partial E}{\partial \theta} \\ -\frac{\partial E}{\partial r} \\ 0 \end{pmatrix} \qquad (13)_u$$

(This holds both inside and outside the plasma.)

The electric field inside the plasma is purely axial and is given by an expression of the form

$$E = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta} \qquad (14)_u$$

where the functions f_m depend upon the nature of the radial dependence of the plasma electron density. In the case where the plasma is collisionless and uniform radially, the f_m reduce to Bessel Functions (J_m for the underdense case and I_m for the overdense case) with argument $\left(rk_o\sqrt{|1-P|}\right)$. For electron density varying radially as $1/r^2$, the f_m are given by Bessel Functions of non-integer order [1, 2], while for more general radial distributions of electron density, a combination of such functions may be applicable [1, 2] or numerical solution of the radial equation may be required [3, 4].

At any rate, the general internal electric field is of the form (14). Moreover, boundedness of the field at the centerline dictates that each f_m involved exactly one undertermined coefficient. Thus there remain, for each θ - harmonic (each m) two undetermined coefficients (c_m and the undetermined coefficient in f_m). The continuity of E (E axial) and H_θ across the plasma-vacuum interface then give precisely the two conditions necessary to fix the undetermined coefficient in f_m and the coefficient c_m in the scattered field. Determination of the c_m completes the description of the scattered field from which, in turn, the scattered energy can be computed.

B. Extension to Axially Nonuniform Plasma

Now the mode of reasoning introduced for the case of an axially varying plasma is as follows:

The form $(12)_u$ in the axially uniform problem suggests that the scattered field may be considered as arising from a super position of line multipoles; that is, multipoles uniformly distributed with respect to z over the centerline of the plasma cylinder. (The representation is, of course, valid only outside the plasma). Associated with $m = 0$ is the line source, with $m = \pm 1$ is the line

dipole, etc.. The strength of each of these multipoles is given by the appropriate coefficient c_m which depends on the details of the radial distribution of electron density. The effects of the radial distribution of electron density (and collision frequency) thus become lumped into a single coefficient c_m for each order m of multipole.

This suggests that, as an approximation for slowly axially varying plasma configurations, a sort of "strip-theory" could be invoked in which the scattered field would continue to be expressed as $(12)_u$ but each c_m would be considered a function of z, this function to be that appropriate to the two-dimensional problem with plasma properties given by those of the actual configuration at the local value of z. This has several obvious deficiencies. First, the resulting field is no longer a solution of the vacuum field equations.

Second, the electric field would persist in being purely axial and the magnetic field, purely transverse; and this is not the true state of affairs. Third, the axial field dependence would be purely that of the plasma or some function thereof and would not represent the axial propagation which actually occurs. In fact, the only conditions really met by the strip-theory solution would be the jump conditions across the vacuum-plasma interface and the radial radiation condition.

Suppose, however, that instead of a simple strip-theory as mentioned above, one invokes a modified form of strip-theory in which the line multipole density is made z-dependent and the field is computed from the resulting z-dependent multipole distributions. That is, the c_m's are interpreted as multipole intensities and these multipole intensities are made z-dependent in accordance with a <u>local</u> or strip-theory value of c_m. Then the total three-dimensional scattered field is calculated as having arisen from this z-dependent line distribution of multipole intensities. The resulting field then (1) satisfies the

vacuum field equations (2) exhibits axial propagation (3) has all three electric vector components (but still lacks an axial magnetic vector component). (4) satisfies the radiation condition. The chief failings are that the magnetic field persists in remaining purely transverse, (as will be seen subsequently) and the interface conditions are violated, the degree of violation depending upon the degree of nonuniformity of the axial plasma dependence. These failings may appear to render the method questionable; however several arguments lend justification to the procedure. First, the solution does reduce to the correct one as the axial dependence of plasma properties approaches zero. Second, the method can be shown to correspond to the first step in an iterative procedure based upon an integral-equation formulation of the problem. Third, the exact solution (in the axially nonuniform plasma case) should be representable by <u>some</u> distribution (or pair of distributions, as will be shown) of line multipoles along the centerline, and we are computing an approximation to the correct distributions.

In the vacuum, outside the cylindrical plasma structure, the field equations (1), (2) reduce to the classical Maxwell's equations for harmonic time dependence.

$$\nabla \times \vec{E} = i\omega\mu_0 \vec{H} \qquad (1)$$

$$\nabla \times \vec{H} = -i\omega\epsilon_0 \vec{E} \qquad (15)$$

Utilizing the ideas of Stratton[5] (p. 393 and ff.) it can be shown that a general solution to these field equations is expressible through the use of two independent scalar quantities φ and χ. (This can even be done inside the plasma when the plasma has purely axial nonuniformity but is independent of x, y or r, θ.) The general solution in terms of scalars φ and χ may be expressed as follows:

$$\vec{E} = \begin{pmatrix} E_r \\ E_\theta \\ E_z \end{pmatrix} = \begin{pmatrix} -\frac{1}{r}\frac{\partial \varphi}{\partial \theta} \\ \frac{\partial \varphi}{\partial r} \\ 0 \end{pmatrix} \qquad \frac{-1}{k_o}\begin{pmatrix} \frac{\partial^2 \chi}{\partial r\,\partial z} \\ \frac{1}{r}\frac{\partial^2 \chi}{\partial \theta\,\partial z} \\ \frac{\partial^2 \chi}{\partial z^2} + k_o^2\,\chi \end{pmatrix} \tag{16}$$

$$\vec{H} = \begin{pmatrix} H_r \\ H_\theta \\ H_z \end{pmatrix} = \frac{k_o}{i\omega\mu_o}\begin{pmatrix} -\frac{1}{r}\frac{\partial \chi}{\partial \theta} \\ \frac{\partial \chi}{\partial r} \\ 0 \end{pmatrix} \quad \frac{-1}{i\omega\mu_o}\begin{pmatrix} \frac{\partial^2 \varphi}{\partial r\,\partial z} \\ \frac{1}{r}\frac{\partial^2 \varphi}{\partial \theta\,\partial z} \\ \frac{\partial^2 \varphi}{\partial z^2} + k_o^2\,\varphi \end{pmatrix} \tag{17}$$

Where φ and χ satisfy reduced wave equations:

$$\Delta\chi + k_o^2\ \chi = 0 \ \text{ and } \Delta\varphi + k_o^2\ \varphi = 0 \tag{18}$$

Obviously the φ - field and χ - field are independent. Morever, utilizing the fact that φ and χ both satisfy the reduced wave equations (18) it is easily seen that the φ - field (16) and (17) satisfies the vacuum field equations (1) and (15) and so does the χ - field. It is not difficult to convince oneself, further, that (16) (17) (18) provide a general solution to the vacuum field equations. This can be done simply through a counting argument on the numbers of boundary conditions necessary to render a problem determined and the number of boundary conditions required to determine completely (and not over-determine) the solutions to the two problems for φ and χ. The two countings agree, indicating that the solution (16) (17) (18) is general.

Next, we seek line multipole solutions to the system (16) (17) (18). As is well-known, the simple source solution to the reduced wave equation in three-space is given, for $e^{-i\omega t}$ time-dependence, by

$$\frac{e^{ik_o\rho}}{\rho} \tag{19}$$

Where ρ is distance between source and field point.

A centerline distribution of sources is then obviously still a solution of the reduced wave equation and is given by

$$\int_{\zeta=-\infty}^{\infty} f(\zeta)\,\frac{e^{ik_o\rho}}{\rho}\,d\zeta \tag{20}$$

where

$$\rho = \sqrt{x^2 + y^2 + (z-\zeta)^2} = \sqrt{r^2 + (z-\zeta)^2} \tag{21}$$

and where $f(\zeta)$ gives the source density and must satisfy such conditions that the integral (20) exists. It is well-known that (20) reduces to a multiple of the symmetric Hankel Function $H_o^{(1)}(k_o r)$ when $f(\zeta)$ is constant, independent of ζ. Letting $(\zeta - z) = r\sinh\tau$,

$$\int_{-\infty}^{\infty} \frac{e^{ik_o\rho}}{\rho}\,d\zeta = \int_{-\infty}^{\infty} e^{ik_o r\cosh\tau}\,d\tau = i\pi\ H_o^{(1)}(k_o r) \tag{22}$$

Thus (20) is an extension of the axially symmetric (simple line source) two-dimensional field $\{H_o^{(1)}(k_o r)\}$ to the z-dependent or three-dimensional case with the density function $f(z)$ remaining to be appropriately determined.

The higher-order multipoles may now be easily derived from the simple source distribution (20). Obviously if $u(x,y,z)$ is a solution of the reduced wave equation, then so is

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} u(x, y, z) \tag{23}$$

Suppose we introduce the notation

$$I_n(x, y, z) = \left(\frac{-1}{k_o}\right)^n \left(\frac{\partial}{\partial x} + \frac{i\partial}{\partial y}\right)^n \frac{1}{i\pi} \int_{-\infty}^{\infty} f_n(\zeta) \frac{e^{ik_o\rho}}{\rho} \, d\zeta \tag{24}$$

$n = 0, 1, 2 \ldots\ldots$

The I_n are all evidently solutions of the reduced wave equation (under suitable integrability restrictions on the $f_n(\zeta)$). Moreover

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) \tag{25}$$

Thus

$$I_n = \left(\frac{-1}{k_o}\right)^n \left[e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)\right]^n \frac{1}{i\pi} \int_{-\infty}^{\infty} f_n(\zeta) \frac{e^{ik_o\rho}}{\rho} \, d\zeta \tag{26}$$

Note that if all the $f_n(\zeta) \equiv 1$ for all n, then

$$I_o = H_o^{(1)}(k_o r) \tag{27}$$

and, using mathematical induction,

$$\underline{\text{if}} \quad I_n = e^{in\theta} H_n^{(1)}(k_o r) \tag{28}$$

then

$$\left.\begin{aligned} I_{n+1} &= \frac{-1}{k_o} e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) e^{in\theta} H_n^{(1)}(k_o r) \\ &= \frac{-1}{k_o} e^{i(n+1)\theta} \left(-\frac{n}{r} H_n^{(1)} + k_o H_n^{(1)'}\right) \\ &= e^{i(n+1)\theta} H_{n+1}^{(1)}(k_o r) \end{aligned}\right\} \tag{29}$$

utilizing the recurrence relations [6] for the Hankel Functions. Thus, in the special case where the density functions are constant, the I_n reduce to the two-dimensional solutions to the reduced wave equation

$$e^{in\theta} H_n^{(1)}(k_o r) \tag{30}$$

Hence, for ζ-dependent density functions f_n, the integrals I_n must represent three-dimensional solutions to the reduced wave equation corresponding to a distribution of an n^{th} order multipole along the centerline, the density being given by the $f_n(\zeta)$.

Similarly, if we introduce

$$\left.\begin{aligned} I_{-n} &= \left(\frac{1}{k_o}\right)^n \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^n \frac{1}{i\pi} \int_{-\infty}^{\infty} f_{-n}(\zeta) \frac{e^{ik_o\rho}}{\rho} \, d\zeta \\ n &= 1, 2, 3 \ldots \\ &= \left(\frac{1}{k_o}\right)^n \left[e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right)\right]^n \frac{1}{i\pi} \int_{-\infty}^{\infty} f_{-n}(\zeta) \frac{e^{ik_o\rho}}{\rho} \, d\zeta \end{aligned}\right\} \tag{31}$$

it follows that when $f_{-n} \equiv 1$ for all n, then

$$I_{-n} = e^{-in\theta} H_{-n}^{(1)}(k_o r) \tag{32}$$

$n = 1, 2, 3 \ldots$

so that the expressions I_{-n} of (31) constitute the extension to three-dimensionality of the negative-index terms in the two-dimensional expansion (12)u.

The expressions (26) and (31) can be put in more convenient form as follows:

$$I_n = \left(-\frac{1}{k_o}\right)^n \left(e^{in\theta}\right)\left(\frac{\partial}{\partial r} - \frac{n-1}{r}\right)\left(\frac{\partial}{\partial r} - \frac{n-2}{r}\right)\cdots\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\left(\frac{\partial}{\partial r} - \frac{0}{r}\right)\frac{1}{i\pi}\int_{-\infty}^{\infty} f_n(\zeta)\frac{e^{ik_o\rho}}{\rho}\,d\zeta$$

$$= \left(\frac{-1}{k_o}\right)^n e^{in\theta}\left(r^{n-1}\frac{\partial}{\partial r}\frac{1}{r^{n-1}}\right)\cdots\left(r^2\frac{\partial}{\partial r}\frac{1}{r^2}\right)\left(r\frac{\partial}{\partial r}\frac{1}{r}\right)\left(\frac{\partial}{\partial r}\right)\frac{1}{i\pi}\int_{-\infty}^{\infty} f_n\frac{e^{ik_o\rho}}{\rho}\,d\zeta$$

$$= \left(\frac{-r}{k_o}\right)^n e^{in\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{1}{i\pi}\int_{-\infty}^{\infty} f_n\frac{e^{ik_o\rho}}{\rho}\,d\zeta \qquad (33)$$

$n = 0, 1, 2 \ldots$

and similarly

$$I_{-n} = \left(\frac{r}{k_o}\right)^n e^{-in\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{1}{i\pi}\int_{-\infty}^{\infty} f_{-n}\frac{e^{ik_o\rho}}{\rho}\,d\zeta \qquad (34)$$

$n = 1, 2, 3 \ldots$

Now the expressions (33) (34) can be given in a still more convenient form in terms of spherical Hankel Functions $h_n^{(1)}(k_o\rho)$ (see, for example, Stratton [5] p. 404 and ff.) These functions may be defined as follows:

$$h_n^{(1)}(k_o\rho) = \sqrt{\frac{\pi}{2k_o\rho}}\; H_{n+\frac{1}{2}}^{(1)}(k_o\rho) \qquad (35)$$

Consequently it is easily seen that

$$h_o^{(1)}(k_o\rho) = \frac{e^{ik_o\rho}}{ik_o\rho} \qquad (36)$$

Therefore, from (33) we see that

$$I_o = \frac{k_o}{\pi}\int_{-\infty}^{\infty} f_o(\zeta)\, h_o^{(1)}(k_o\rho)\,d\zeta \qquad (37)$$

Now we resort again to mathematical induction:

If $\quad I_n = e^{in\theta}\dfrac{k_o}{\pi}\displaystyle\int_{-\infty}^{\infty}\left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\, f_n(\zeta)\,d\zeta \qquad (38)$

Then, by (33)

$$I_{n+1} = -\frac{1}{k_o}e^{i(n+1)\theta}\left(\frac{\partial}{\partial r} - \frac{n}{r}\right)\frac{k_o}{\pi}\int_{-\infty}^{\infty}\left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho) f_{n+1}(\zeta)\,d\zeta \qquad (39)$$

$$= -\frac{1}{\pi}e^{i(n+1)\theta}\int_{-\infty}^{\infty} f_{n+1}(\zeta)\left\{-n\frac{r^{n-1}}{\rho^n}h_n^{(1)} + \frac{nr^{n-1}}{\rho^n}h_n^{(1)} - \frac{nr^{n+1}}{\rho^{n+2}}h_n^{(1)} + k_o\frac{r^{n+1}}{\rho^{n+1}}h_n^{(1)'}\right\}d\zeta \qquad (40)$$

but [5], the spherical Hankel functions satisfy the recurrence relation:

$$h_n^{(1)'}(k_o\rho) - \frac{n}{k_o\rho}h_n^{(1)}(k_o\rho) = -h_{n+1}^{(1)}(k_o\rho) \qquad (41)$$

Hence,

$$I_{n+1} = -\frac{1}{\pi}e^{i(n+1)\theta}\int_{-\infty}^{\infty} f_{n+1}(\zeta)\left\{\frac{-nr^{n+1}}{\rho^{n+2}}h_n^{(1)} + k_o\frac{r^{n+1}}{\rho^{n+1}}\left[\frac{n}{k_o\rho}h_n^{(1)} - h_{n+1}\right]\right\}d\zeta$$

$$= e^{i(n+1)\theta}\frac{k_o}{\pi}\int_{-\infty}^{\infty} f_{n+1}(\zeta)\left(\frac{r}{\rho}\right)^{n+1} h_{n+1}^{(1)}(k_o\rho)\,d\zeta \qquad (42)$$

and it is proven that (38) holds:

$$I_n = e^{in\theta}\ \frac{k_o}{\pi} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\ f_n(\zeta)\ d\zeta \tag{38}$$

$$n = 0, 1, 2, \ldots\ldots$$

By inspection of (34) we see that

$$I_{-n} = e^{-in\theta}\ (-1)^n\ \frac{k_o}{\pi} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\ f_{-n}(\zeta)\ d\zeta \tag{43}$$

$$n = 1, 2, 3, \ldots\ldots$$

Now the I_n, I_{-n}, given by (38) and (43) constitute three dimensional solutions to the reduced wave equation constructed from linear distributions of multipoles of order n, the $f_n(\zeta)$ giving the line density of the distribution. Recalling the system (16), (17), (18), giving a general solution to the vacuum field equations, we see that two solutions, $\underline{\varphi}$ and $\underline{\chi}$, of the reduced wave equation are required. These solutions are then introduced as follows:

$$\left.\begin{aligned} \varphi &= \sum_{-\infty}^{\infty} e^{in\theta}\ \varphi_n(r,z) \\ \chi &= \sum_{-\infty}^{\infty} e^{in\theta}\ \chi_n(r,z) \end{aligned}\right\} \tag{44}$$

where

$$\left.\begin{aligned} \varphi_n &= \frac{k_o}{\pi} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\ f_n(\zeta)\ d\zeta & n &= 0,1,2,3\ldots \\ \varphi_{-n} &= (-1)^n \frac{k_o}{\pi} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\ f_{-n}(\zeta)\ d\zeta & n &= 1,2,3\ldots \\ \chi_n &= \frac{k_o}{\pi} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\ g_n(\zeta)\ d\zeta & n &= 0,1,2,3\ldots \\ \chi_{-n} &= (-1)^n \frac{k_o}{\pi} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^n h_n^{(1)}(k_o\rho)\ g_{-n}(\zeta)\ d\zeta & n &= 1,2,3\ldots \end{aligned}\right\} \tag{45a}$$

or

$$\begin{Bmatrix} \varphi_n \\ \chi_n \end{Bmatrix} = \frac{k_o}{\pi} (-1)^{\frac{|n|-n}{2}} \int_{-\infty}^{\infty} \left(\frac{r}{\rho}\right)^{|n|} h_{|n|}^{(1)}(k_o\rho) \begin{Bmatrix} f_n(\zeta) \\ g_n(\zeta) \end{Bmatrix} d\zeta \qquad \text{all n} \tag{45b}$$

Then the corresponding vacuum scattered fields are given by

$$\vec{E}_{scat.} = \sum_{-\infty}^{\infty} e^{in\theta} \left\{ \begin{pmatrix} -\frac{in}{r}\varphi_n \\ \frac{\partial \varphi_n}{\partial r} \\ 0 \end{pmatrix} - \frac{1}{k_o} \begin{pmatrix} \frac{\partial^2 \chi_n}{\partial r\, \partial z} \\ \frac{in}{r} \frac{\partial \chi_n}{\partial z} \\ \frac{\partial^2 \chi_n}{\partial z^2} + k_o^2\, \chi_n \end{pmatrix} \right\} \tag{46}$$

$$\vec{H}_{scat.} = \sum_{-\infty}^{\infty} e^{in\theta} \left\{ \frac{k_o}{i\omega\mu_o} \begin{pmatrix} -\frac{in}{r}\chi_n \\ \frac{\partial \chi_n}{\partial r} \\ 0 \end{pmatrix} - \frac{1}{i\omega\mu_o} \begin{pmatrix} \frac{\partial^2 \varphi_n}{\partial r\, \partial z} \\ \frac{in}{r} \frac{\partial \varphi_n}{\partial z} \\ \frac{\partial^2 \varphi_n}{\partial z^2} + k_o^2\, \varphi_n \end{pmatrix} \right\} \tag{47}$$

Now expressions (44) through (47) have the capability of providing the exact vacuum scattered field if the multipole densities f_n, g_n could be computed exactly. Instead of attempting this exact computation of the densities, we resort, at this point, to an approximation based on a strip theory, the approximation becoming exact as the axial nonuniformity of the plasma reduces to zero. Inspection of $(12)_u$, $(13)_u$, and (28) or (30) shows that, in the limit of an axially uniform plasma, the $\bar{E}$-field is purely axial, the $\bar{H}$-field is transverse ($H_z \equiv 0$) and therefore

$$\left.\begin{aligned} f_n &\equiv 0, \text{ all } n \\ g_n &\equiv \text{constant} = -\frac{c_n}{k_o}, \text{ all } n \end{aligned}\right\} \quad (48)$$

where the c_n are computed in accordance with the discussion following eq. $(14)_u$ and contain all the influences of the radial distribution of plasma properties. The type of strip theory proposed herein then goes as follows: for the axially nonuniform plasma, provided that the nonuniformity is gradual (z-gradients of plasma properties are small) the functions f_n and g_n are to be given by

$$\left.\begin{aligned} f_n &\equiv 0 \qquad \text{all } n \\ g_n(\zeta) &= -\frac{c_n(\zeta)}{k_o} \end{aligned}\right\} \quad (49)$$

where $c_n(\zeta)$ is the value of the 2-dimensional c_n that corresponds to the plasma properties at ζ. The scattered field is then given by (46) and (47) with the φ_n all vanishing, with the χ_n given by the last two of expressions (45), and with the g_n given by (49). The resulting $\vec{E}$-field will then have all three components, but the $\vec{H}$-field will remain purely transverse, as in the axially uniform case. Moreover the resulting $\vec{E}$ and $\vec{H}$ scattered fields will be solutions of the vacuum field

equations and will satisfy the radiation condition since they are synthesized from line distributions of miltipoles. Having the scattered field, obtained by the preceding approximate process, and knowing, as we do, the incident field, the total vacuum field is described. Then the internal (plasma) field is completely determined by the tangential (E_z and E_θ) components of vacuum total electric field and the requirement of continuity of these components across the plasma-vacuum interface. If we were to compute the internal field and check the continuity of the tangential components of the magnetic field across the interface we would find that, if the scattered field is exactly correct, then tangential magnetic field continuity will be realized; and if the scattered field is not exact, then tangential magnetic field continuity will be violated. It is in this violation that the error in our approximation appears, the error, of course, going to zero as axial plasma nonuniformity is reduced to zero.

C. Integral Equation Method

There is an alternative method of arriving at the approximation developed in the preceeding section. This involves the formulation of the scattering problem as a vector integral equation. Since this alternative formulation sheds some further light on the implications of the approximation introduced in the multipole method, is is worthwhile to reproduce the integral equation formulation at this point.

Stratton and Chu[5] have developed a vector analog of Green's theorem which expresses the electric and magnetic fields in terms of surface integrals involving surface values of the fields themselves and volume integrals involving spatial distributions of current and charge. For our purposes, we assume the plasma

electron density to be "smeared out" at the plasma-vacuum interface. That is, we assume that the electron density changes continuously (together with its gradient) over a thin layer from zero in the vacuum to its edge value in the plasma. This eliminates the plasma-vacuum interface surface from the problem and allows us to concern ourselves only with volume integrals plus the limiting surface integrals on a surface which is extended to infinity. In this case, Stratton and Chu derive the result

$$\vec{E}(p) = \frac{1}{4\pi}\int_{V_S}\left[i\omega\mu_o \, \vec{J}(g)\frac{e^{ik_o\rho}}{\rho} + \frac{1}{\epsilon_o}\tilde{q}(g)\nabla_g \frac{e^{ik_o\rho}}{\rho} \right] dV_g + \int_S \tag{50}$$

$$\vec{H}(p) = \frac{1}{4\pi}\int_{V_S}\vec{J}(g)\times\nabla_g\left(\frac{e^{ik_o\rho}}{\rho}\right) dV_g + \int_S \tag{51}$$

(where g is integration-variable point, p is field-variable point, ρ is distance between p and g, and $\int_S$ is a surface integral over bounding surface S of volume V_S) as a solution to the system

$$\begin{aligned} \nabla\times\vec{E} &= i\omega\mu_o\vec{H} \\ \nabla\times\vec{H} &= -i\omega\epsilon_o\vec{E}+\vec{J} \\ \epsilon_o\nabla\cdot\vec{E} &= \tilde{q} = \text{charge density} \end{aligned} \tag{52}$$

Comparison of (52) with (1) and (4) of the present report shows that we may interpret the plasma in terms of equivalent current and charge densities.

$$\vec{J} = i\omega\epsilon_o\frac{\omega_p^2}{\omega^2+\nu^2}\left(1-\frac{i\nu}{\omega}\right)\vec{E}$$

$$= i\omega\epsilon_o P(r,z)\vec{E} \tag{53}$$

$$\text{with } P = \frac{\omega_p^2}{\omega^2+\nu^2}\left(1-\frac{i\nu}{\omega}\right)$$

and

$$\tilde{q} = \epsilon_o\nabla\cdot\vec{E} = \epsilon_o\nabla\cdot(P\vec{E})$$

$$\text{(from (4) and (53))} \tag{54}$$

Thus, keeping in mind the smeared out plasma interface we may write for the plasma problem, the analog of the Stratton and Chu relations:

$$\vec{E}(p) = \frac{1}{4\pi}\int_{V_S}\left\{-k_o^2 P(g)\vec{E}(g)\frac{e^{ik_o\rho}}{\rho} + \nabla_g\cdot\vec{E}(g)\nabla_g\frac{e^{ik_o\rho}}{\rho}\right\} dV_g + \int_S \tag{55}$$

$$\vec{H}(p) = \frac{1}{4\pi}\int_{V_S} i\omega\epsilon_o P(g)\vec{E}(g)\times\nabla_g\left(\frac{e^{ik_o\rho}}{\rho}\right) dV_g + \int_S \tag{56}$$

Now if $\vec{E}$ and $\vec{H}$ represent the total field, then the surface integrals $\int_S$ should be such as to vanish as the surface S becomes indefinitely large except for the contribution of the incident wave which violates the radiation condition. This incident wave should contribute, through the surface integral, exactly

the quantity $\vec{E}_{inc.}$ to (55) and $\vec{H}_{inc.}$ to (56). Thus we have, as the surface S recedes to infinity,

$$\vec{E}_{total}^{(p)} = \vec{E}_{inc}^{(p)} + \frac{1}{4\pi} \int_{\substack{\text{plasma}\\ \text{volume}}} \left\{ -k_o^2 P(g)\vec{E}(g) \frac{e^{ik_o\rho}}{\rho} + \nabla_g \cdot \vec{E}(g) \nabla_g \frac{e^{ik_o\rho}}{\rho} \right\} dV_g \qquad (57)$$

$$\vec{H}_{total}(p) = \vec{H}_{inc.}^{(p)} + \frac{1}{4\pi} \int_{\substack{\text{plasma}\\ \text{volume}}} i\omega\epsilon_o P(g)\, \vec{E}(g) \times \nabla_g \frac{e^{ik_o\rho}}{\rho} \, dV_g \qquad (58)$$

Actually (58) follows directly from (1) and (57).

Eq. (57) appears to be a vector integro-differential equation for the electric field but can be reduced to an ordinary vector integral equation by making use of (54). From Eq. (54) we see that

$$\nabla \cdot \vec{E} = \nabla P \cdot \vec{E} + P \nabla \cdot \vec{E} \qquad (59)$$

so that

$$\nabla \cdot \vec{E} = \frac{\nabla P \cdot \vec{E}}{1 - P} \qquad (60)$$

Therefore we obtain a vector integral equation for the electric field:

$$\vec{E}(p) = \vec{E}_{inc}(p) + \frac{1}{4\pi} \int_{\substack{\text{plasma}\\ \text{volume}}} \left\{ -k_o^2 P(g)\, \vec{E}(g) \frac{e^{ik_o\rho}}{\rho} + \frac{\nabla_g P(g) \cdot E(g)}{1 - P(g)} \nabla_g \frac{e^{ik_o\rho}}{\rho} \right\} dV_g \qquad (61)$$

Now the field point p in (61) may lie inside or outside the plasma. The incident field $\vec{E}_{inc}$ is given everywhere by

$$\begin{bmatrix} 0 \\ 0 \\ E_o \end{bmatrix} e^{-ik_o x} \qquad (62)$$

It will be shown that (61) can be related, for the case where point P lies <u>outside</u> the plasma, to the multipole distribution formulation of the preceding section; but first a few remarks are in order concerning the system (61) itself. If (61) is to be solved as an integral equation system then, of course, points p <u>inside</u> the plasma must be considered. Furthermore, in the limit where the plasma is axially uniform, P depends only upon radius r and not upon z. Then it is evident from inspection of (61) and (62) that a purely axial electric field will satisfy (61) and that the second term disappears from the integrand. It is equally evident, however, that as small axial gradients in plasma properties occur ($\frac{\partial P}{\partial z}$ small, non-zero) then all three components must appear in $\vec{E}$, though the transverse (r, θ) components may be small. Correspondingly, all three components will appear in $\vec{H}$, though the axial component may be small.

The relation between eq. (61), for points p outside the plasma, and the line multipole distribution approach may be seen through the following development: the free-space Green's function or simple-pole solution for the reduced wave equation $\frac{e^{ik_o\rho}}{\rho}$ can be expanded in spherical wave functions.[7]

$$\frac{e^{ik_o\rho}}{\rho} = ik_o \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos\left[m(\theta_p - \theta_g)\right]$$

$$P_n^m(\cos\varphi_p)\; P_n^m(\cos\varphi_g) \begin{Bmatrix} j_n(k_o R_g)\, h_n^{(1)}(k_o R_p) \\ h_n^{(1)}(k_o R_g)\, j_n(k_o R_p) \end{Bmatrix}$$

$$\text{for} \begin{Bmatrix} R_g < R_p \\ R_g > R_p \end{Bmatrix} \qquad (63)$$

where $\epsilon = 1$, $m = 0$

$\epsilon = 2$, $m > 0$

j_n are the ordinary spherical Bessel functions[5,7]

$h_n^{(1)}$ are the spherical Hankel functions

P_n^m are the associated Lengendre Polynomials[5,7]

R is radius in spherical coordinates

$(\,)_p$, $(\,)_g$ represent field point and source point, respectively

ρ is distance between points p and g

θ is longitude angle

φ is colatitude angle

Now suppose we choose, as origin for the spherical coordinate system appropriate to expression (63), the point ($x = o$, $y = o$, $z = z_g$) i.e. the point on the plasma center line in the (x, y) plane of the source point g. Then

$$\left.\begin{aligned} R_g &= r_g = \text{cylindrical radius} \\ \varphi_g &= \pi/2 \\ R_p &= \sqrt{(z_p - z_g)^2 + r_p^2} \\ \varphi_p &= \arctan\frac{r_p}{z_p - z_g} = \arcsin\frac{r_p}{R_p} \end{aligned}\right\} \qquad (64)$$

and for points g inside the plasma and p outside the plasma, $R_p > R_g$. With this choice, the expression (63) reduces to

$$\frac{e^{ik_o\rho}}{\rho} = ik_o \sum_{n=0}^{\infty} (2n+1) \sum_{m=0}^{n} \epsilon_m \frac{(n-m)!}{(n+m)!} \cos m(\theta_p - \theta_g)\, P_n^m(o)\, P_n^m(\cos\varphi_g)$$

$$j_n(k_o r_g)\, h_n^{(1)}(k_o R_p) \qquad (65)$$

Now eq. (61) in its present form must be interpreted in terms of Cartesian field components E_x E_y E_z rather than cylindrical components. In order to convert to a form suitable for cylindrical components we make linear combinations of the x and y components of (61) to form E_r and E_θ at the field point p and, at the same time, express the internal $E_x(g)$ $E_y(g)$ as linear combinations of $E_r(g)$, $E_\theta(g)$. Also we note that the gradient ∇_g in the second term of the integrand may be replaced by $-\nabla_p$ (since it operates on ρ) and, if desired, taken outside the integral. The resulting cylindrical form is

$$\begin{pmatrix} E_r(p) \\ E_\theta(p) \\ E_z(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ E_{z_{inc}}(p) \end{pmatrix} - \frac{1}{4\pi} \int_{\substack{\text{plasma}\\ \text{volume}}} \left\{ k_o^2 \, P(g) \begin{pmatrix} E_r(g) \cos(\theta_g - \theta_p) - E_\theta(g) \sin(\theta_g - \theta_p) \\ E_\theta(g) \cos(\theta_g - \theta_p) + E_r(g) \sin(\theta_g - \theta_p) \\ E_z(g) \end{pmatrix} \right.$$

$$\left. + \frac{\nabla_g P(g) \cdot \vec{E}(g)}{1 - P(g)} \begin{pmatrix} \frac{\partial}{\partial r_p} \\ \frac{1}{r_p} \frac{\partial}{\partial \theta_p} \\ \frac{\partial}{\partial z_p} \end{pmatrix} \right\} \frac{e^{ik_o \rho}}{\rho} \, dV_g \qquad (66)$$

Expanding the electric field components in Fourier (with respect to θ) components, noting that ρ depends on θ_p, θ_g only through the combination $\theta_g - \theta_p$, defining

$$\theta_o = \theta_g - \theta_p \qquad (67)$$

and introducting expression (65) for $\frac{e^{ik_o \rho}}{\rho}$, there results, for each (m^{th}) Fourier component of the electric field vector:

$$\begin{pmatrix} E_r^m(p) \\ E_\theta^m(p) \\ E_z^m(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ E^m_{z_{inc}}(p) \end{pmatrix} - \frac{ik_o}{4\pi} \int_{-\infty}^{\infty} dz_g \int_0^{r_{\text{plasma boundary}}} r_g \, dr_g \int_0^{2\pi} e^{im\theta_o} \, d\theta_o \left\{ \text{(continued)} \right.$$

$$\left\{ k_o^2 \, P(r_g, z_g) \begin{pmatrix} E_r^m(g) \cos\theta_o - E_\theta^m(g) \sin\theta_o \\ E_\theta^m(g) \cos\theta_o + E_r^m(g) \sin\theta_o \\ E_z^m(g) \end{pmatrix} + \frac{\nabla_g P(g) \cdot \vec{E}^m(g)}{1 - P(g)} \begin{pmatrix} \frac{\partial}{\partial r_p} \\ \frac{1}{r_p} \frac{\partial}{\partial \theta_o} \\ \frac{\partial}{\partial z_p} \end{pmatrix} \right\}$$

$$\sum_{n=0}^{\infty} (2n+1) \sum_{\nu=0}^{n} \epsilon_\nu \frac{(n-\nu)!}{(n+\nu)!} \cos \nu\theta_o \, P_n^\nu(o) \, P_n^\nu(\cos\varphi_p) \, j_n(k_o r_g) \, h_n^{(1)}(k_o R_p)$$

$$(68)$$

It should be noted that an integration by parts, with respect to θ_o, is equivalent to a replacement of the gradient $\left(-\frac{1}{r_p} \frac{\partial}{\partial \theta_o} \right)$ by $\left(+ \frac{im}{r_p} \right)$ in the second component of the second term of the integrand.

We will content ourselves with exhibiting the correspondence between (68) and the multipole formulation for the z-component E_z of the field. The argument for the E_r E_θ components is slightly more cumbersome, but proceeds similarly. Performing the θ_o integration and interchanging the order of ν and n summations, we find

$$E_z^m(p) = E^m_{z_{inc}}(p) - \frac{ik_o}{2} \int_{z_g = -\infty}^{\infty} dz_g \int_0^{r_{\text{plasma boundary}}} r_g \, dr_g \left\{ k_o^2 \, P(r_g z_g) E_z^m(g) \right.$$

$$\left. + \frac{\nabla_g P(g) \cdot \vec{E}^m(g)}{1 - P(g)} \begin{bmatrix} \frac{\partial}{\partial z_p} \\ \text{or} \\ \frac{-\partial}{\partial z_g} \end{bmatrix} \right\} \sum_{n=|m|}^{\infty} (2n+1) \frac{(n-|m|)!}{(n+|m|)!} P_n^{|m|}(o) \, P_n^{|m|}(\cos\varphi_p) \, j_n(k_o r_g) \, h_n^{(1)}(k_o R_p)$$

$$(69)$$

Now, by using the recurrence relations[5] satisfied by the spherical Bessel functions and by the associated Legendre polynomials, together with the geometric relations given in eq. (64), we find that

$$\frac{\partial}{\partial z_g} P_n^m(\cos\varphi_p) h_n^{(1)}(k_o R_p) = k_o \left\{ \frac{P_{n+1}^m h_{n+1}^{(1)}(n-m+1) - P_{n-1}^m h_{n-1}^{(1)}(n+m)}{2n+1} \right\} \tag{70}$$

Moreover, the associated Legendre polynomials P_n^m cut off or vanish when $m > n$. Thus, any expression of the form

$$P_n^{|m|}(\cos\varphi_p)\; h_n^{(1)}(k_o R_p) \quad , \; n \geq |m|$$

is obtainable with a combination of operations of the form

$$\frac{\partial^\nu}{\partial z_g^\nu}$$

upon $P_{|m|}^{|m|}(\cos\varphi_p)\; h_{|m|}^{(1)}(k_o R_p)$. That is, the first partial leads to

$$P_{|m|+1}^{|m|}\; h_{|m|+1}^{(1)} \; .$$

The second partial leads to

$$P_{|m|+2}^{|m|}\; h_{|m|+2}^{(1)} \text{ and } P_{|m|}^{|m|}\; h_{|m|}^{(1)}$$

The third partial leads to

$$P_{|m|+3}^{|m|}\; h_{|m|+3}^{(1)} \;,\; P_{|m|+1}^{|m|}\; h_{|m|+1}^{(1)}$$

etc.

Thus all terms in (69) of the form

$$P_n^{|m|}(\cos\varphi_p)\; h_n^{(1)}(k_o R_p) \qquad n \geq |m|$$

are expressible as linear combinations of operations of the form

$$\frac{\partial^\nu}{\partial z_g^\nu}$$

upon a single term of the form $P_{|m|}^{|m|}\; h_{|m|}^{(1)}$.

Utilizing this result, and performing the integration with respect to r_g, the expression (69) may be put in the symbolic form:

$$E_z^m(p) = E_{z_{inc}}^m(p) + \int_{z_g=-\infty}^{\infty} dz_g \sum_{n=0}^{\infty} A_n(z_g) \frac{\partial^n}{\partial z_g^n} P_{|m|}^{|m|}(\cos\varphi_p) h_{|m|}^{(1)}(k_o R_p) \tag{71}$$

Then a series of integrations by parts, assuming suitable behavior at $z \to \pm\infty$, permits this to be written in the still more elementary form

$$\left.\begin{aligned} E_z^m(p) &= E_{z_{inc.}}^m(p) + \int_{z_g=-\infty}^{\infty} dz_g\; F(z_g)\; P_{|m|}^{|m|}(\cos\varphi_p) h_{|m|}^{(1)}(k_o R_p) \\ &= E_{z_{inc.}}^m(p) + \int_{z_g=-\infty}^{\infty} dz_g\; F(z_g) \frac{(2n)!}{2^n n!} \left(\frac{r_p}{R_p}\right)^{|m|} h_{|m|}^{(1)}(k_o R_p) \end{aligned}\right\} \tag{72}$$

$$\text{where } F(z_g) = \sum_{n=0}^{\infty} (-1)^n \frac{\partial^n}{\partial z_g^n} A_n(z_g) \tag{73}$$

and where use has been made of the fact that

$$P_m^m(\cos\varphi_p) = \frac{(2n)!}{2^n n!} (\sin\varphi_p)^m \tag{74}$$

$$\text{and} \quad \sin\varphi_p = {}^{r}p/R_p$$

Now, of course, the function $F(z_g)$ is not known and contains the influence of the field $\vec{E}$ itself inside the plasma, but nevertheless the expression (72) shows the connection between the integral equation and multipole methods. It is easily seen that (72) is strictly equivalent to the multipole form for E_z as expressed through relations (45) and (46). (Note that in (45) (46) the required operation $\frac{\partial^2}{\partial z^2}$ on χ_n acts only on ρ, which is equal to R_p, and hence may be replaced by $\frac{\partial^2}{\partial \zeta^2}$ which, in turn, permits integration by parts and a re-interpretation of the density function g_n). Thus the integral equation and the multipole formulation have been shown (at least for the z-component of the electric field) to be equivalent for the exterior of the plasma, i.e. for the scattered field. The arguments for the r- and θ - components are similar but more tedious.

Now the strip-theory method of choosing the functions f_n and g_n in the multipole formulation may be interpreted as the first step in an iterative solution of the integral equation formulation. It is to be recalled that f_n and g_n, interpreted as line multipole distribution densities in the multipole formulation, were to be computed in accordance with a strip theory whereby they were to be related to the values of the 2-dimensional coefficients c_n that correspond to the

local (same value of z) plasma properties. This has the following interpretation in the integral equation formulation of eq. (61). If we insert, as a first trial or approximation in the integrand of (61) the values of $\vec{E}(g)$ which correspond to the two dimensional solutionappropriate to the plasma properties at the local value of z_g, then the next approximation, given by $\vec{E}(p)$ in (61) corresponds to our multipole-strip-theory approximation. In other words the multipole-strip-theory approximation is equivalent to the result of one iteration on the integral equation formulation, starting from a first trial based on the simplest strip theory (2-dimensional results based on the local-z plasma properties).

The question that arises immediately is; will an iterative process applied to the integral equation (61) be convergent? Evidently, for a dilute enough plasma (P small everywhere) the iteration will converge. In fact, the first iteration corresponding to an initial trial of $\vec{E} = \vec{E}_{incident}$, is precisely the "electron-scattering" solution, in which each electron is presumed to act as though influenced only by the incident field and not by the fields scattered by the other electrons. When the plasma is dense, on the other hand, an iterative procedure will probably diverge. However it should be noted that (1) we are starting from an initial trial which is not the incident wave but, rather, is the simple strip theory field; (2) we are restricting the plasma to have weak axial dependence; and (3) the first iteration (and the zeroth, as well) is exact as the axial dependence goes to zero. In effect, we are relying upon these considerations to lend justification to the strip-theory approximation proposed herein.

III. SCATTERED ENERGY AND BACK SCATTERING

The scattered energy flow is calculated by means of the Poynting vector $\vec{S}_{scat.}$ which gives the intensity of energy flow of the scattered field. For harmonic time dependence, the mean intensity of energy flow in the scattered field is given by

$$\vec{S}_{scat.} = \frac{1}{2} R_e \left(\vec{E}_{scat.} \times \vec{H}^*_{scat.} \right) \qquad (75)$$

where (*) denotes complex conjugate. Hence the radial component of scattered energy flow is given by

$$S_{r_{scat.}} = \frac{1}{2} R_e \left(E_\theta H_z^* - E_z H_\theta^* \right) \qquad (76)$$

In our strip theory approximation, since the φ_n are all zero, H_z is zero and

$$S_{r_{scat.}} = \frac{-1}{2} R_e \{ E_z H_\theta^* \}$$
$$= \frac{-1}{2} R_e \left\{ \sum_{n=-\infty}^{\infty} e^{in\theta} \left(\frac{\partial^2 \chi_n}{\partial z^2} + k_o^2 \chi_n \right) \frac{1}{i\omega\mu_o} \sum_{m=-\infty}^{\infty} e^{im\theta} \frac{\partial \chi_m^*}{\partial_r} \right\} \qquad (77)$$

However, since the χ_m are solutions of the reduced wave equation with $e^{im\theta}$-type θdependence, therefore the χ_m satisfy

$$\chi_{m_{zz}} + k_o^2 \chi_m + \chi_{m_{rr}} + \frac{1}{r} \chi_{m_r} - \frac{m^2}{r^2} \chi_m = 0 \qquad (78)$$

Thus, (79) may be replaced by

$$S_{r_{scat.}} = -\frac{1}{2} R_e \left\{ \frac{1}{i\omega\mu_o} \sum_{n=-\infty}^{\infty} e^{in\theta} \left(\frac{n^2}{r^2}\chi_n - \frac{\partial^2 \chi_n}{\partial r^2} - \frac{1}{r} \frac{\partial \chi_n}{\partial r} \right) \sum_{m=-\infty}^{\infty} e^{-im\theta} \frac{\partial \chi_m^*}{\partial r} \right\} \qquad (79)$$

since r is real, the last term in the first summation may be dropped, contributing as it does only a pure imaginary quantity to the product, and the final expression for $S_{r_{scat.}}$ becomes

$$S_{r_{scat.}} = -\frac{1}{2} R_e \left\{ \frac{1}{i\omega\mu_o} \sum_{n=-\infty}^{\infty} e^{in\theta} \left(\frac{n^2 \chi_n}{r^2} - \frac{\partial^2 \chi_n}{\partial_r^2} \right) \sum_{m=-\infty}^{\infty} e^{-im\theta} \frac{\partial \chi_m^*}{\partial_r} \right\} \qquad (80)$$

Now χ_n and χ_m are given by (45). However, we are actually interested only in the far field ($k_o r >> 1$) scattered energy so that only large values of $k_o r$ are pertinent. Thus we replace the spherical Hankel functions in (45) by their asymptotic representations:

$$\left. \begin{aligned} h_n^{(1)}(k_o\rho) &\approx \sqrt{\frac{\pi}{2k_o\rho}} \sqrt{\frac{2}{\pi k_o \rho}}\, e^{i\left(k_o\rho - \frac{\pi}{4} - \frac{\pi}{2}(n+\frac{1}{2})\right)} \\ &\approx \frac{-i}{k_o\rho}\, e^{i\left(k_o\rho - \frac{n\pi}{2}\right)} \end{aligned} \right\} \qquad (81)$$

$$\left. \begin{aligned} &\text{Thus } \chi_n \approx \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{r^n}{(\rho)^{n+1}} e^{ik_o\rho} e^{-in\pi/2} g_n(\zeta)\, d\zeta \\ &\qquad n = 0, 1, 2 - - -, \\ &\chi_{-n} \approx \frac{(-1)}{i\pi} \int_{-\infty}^{\infty} \frac{r^n}{(\rho)^{n+1}} e^{ik_o\rho} e^{-in\pi/2} g_{-n}(\zeta)\, d\zeta \\ &\qquad n = 1, 2, 3, - - -, \end{aligned} \right\} \qquad (82)$$

In the far field $(r \to \infty)$

$$\frac{n^2}{r^2}\chi_n - \frac{\partial^2 \chi_n}{\partial r^2} \approx \frac{-\partial^2}{\partial r^2}{}_n \approx \frac{k_o^2}{i\pi} \int_{-\infty}^{\infty} \frac{r^{n+2}}{\rho^{n+3}} e^{ik_o\rho} e^{-in\pi/2} g_n(\zeta)\, d\zeta$$
$$n = 0, 1, 2, 3, - - -, \qquad (83)$$
$$\frac{n^2}{r^2}\chi_{-n} - \frac{\partial^2 \chi_{-n}}{\partial_r^2} \approx \frac{(-1)^n k_o^2}{i\pi} \int_{-\infty}^{\infty} \frac{r^{n+2}}{\rho^{n+3}} e^{ik_o\rho} e^{-in\pi/2} g_{-n}(\zeta)\, d\zeta$$

These can be put into a single form; valid for all n:

$$\frac{n^2}{r^2} X_n - \frac{\partial^2 X_n}{\partial_r{}^2} \approx \frac{k_o{}^2}{i\pi} e^{-in\pi/2} \int_{-\infty}^{\infty} \frac{r^{|n|+2}}{\rho^{|n|+3}} e^{ik_o\rho} g_n(\zeta)\, d\zeta \tag{84}$$

all n

In a similar manner, we find that

$$\frac{\partial X_m^*}{\partial r} \approx \frac{k_o}{\pi} e^{im\pi/2} \int_{-\infty}^{\infty} \frac{r^{|m|+1}}{\rho^{|m|+2}} e^{-ik_o\rho} g_m^*(\zeta)\, d\zeta \tag{85}$$

all m

Thus the radial component of scattered energy flow in the far field is given by:

$$S_{r_{scat.}} \approx \frac{-1}{2} R_e \left\{ \frac{1}{i\omega\mu_o} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-in\pi/2} \frac{k_o{}^2}{i\pi} \int_{-\infty}^{\infty} \frac{r^{|n|+2}}{\rho^{|n|+3}} e^{ik_o\rho} g_n(\zeta) d\zeta \right.$$

$$\left. \sum_{m=-\infty}^{\infty} e^{-im\theta} \frac{k_o}{\pi} e^{im\pi/2} \int_{-\infty}^{\infty} \frac{r^{|m|+1}}{\rho^{|m|+2}} e^{-ik_o\rho} g_m^*(\zeta)\, d\zeta \right\} \tag{86}$$

The back-scattering, or energy scattered back in the x direction from which the incident wave originates is given by:

$$S_r = S_{r_{scat.}} \Big|_{\theta=0} \approx \frac{k_o{}^3}{2\pi^2 \omega \mu_o} R_e \left\{ \sum_{m=-\infty}^{\infty} e^{-in\pi/2} \int_{-\infty}^{\infty} \frac{r^{|n|+2}}{\rho^{|n|+3}} e^{ik_o\rho} g_n(\zeta)\, d\zeta \right.$$

back scat.

$$\left. \sum_{m=-\infty}^{\infty} e^{im\pi/2} \int_{-\infty}^{\infty} \frac{r^{|m|+1}}{\rho^{|m|+2}} e^{-ik_o\rho} g_m^*(\zeta) d\zeta \right\} \tag{87}$$

$$\text{with } \rho = \sqrt{r_p{}^2 + \left(z_p - \zeta\right)^2}$$

We may define a form of back scattering cross section length σ_B as follows:

$$\sigma_B = \frac{2\pi r\, S_{r_{scat}}\big|_{\theta=o}}{\text{Incident energy flow/unit area}}$$

$$= \frac{2\pi r\, S_{r_{scat}}\big|_{\theta=o}}{E_o{}^2 k_o / 2\omega\mu_o} \tag{88}$$

Combining (87) and (88) we find

$$\sigma_B(r_p, z_p) = \frac{2k_o{}^2 r}{\pi E_o{}^2} R_e \left\{ \sum_{n=-\infty}^{\infty} e^{-in\pi/2} \int_{-\infty}^{\infty} \frac{r^{|n|+2}}{\rho^{|n|+3}} e^{ik_o\rho} g_n(\zeta)\, d\zeta \sum_{m=-\infty}^{\infty} e^{im\pi/2} \right.$$

$$\left. \int_{-\infty}^{\infty} \frac{r^{|m|+1}}{\rho^{|m|+2}} e^{-ik_o\rho} g_m^*(\zeta) d\zeta \right\} \tag{89}$$

We now insert the result of the strip theory approximations

$$g_n(\zeta) = -\frac{c_n(\zeta)}{k_o} \tag{49}$$

and, noting that the c_n will be proportional to the n^{th} Fourier coefficient of the incident wave, we define coefficients γ_n by

$$c_n(\zeta) = \gamma_n(\zeta)\, E_o e^{-in\pi/2} \tag{90}$$

In terms of the γ_n, the back scattering cross section σ_B becomes:

$$\sigma_B(r_p, z_p) = \frac{2r}{\pi} R_e \left\{ \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{r^{|n|+2}}{\rho^{|n|+3}} e^{ik_o\rho} \gamma_n(\zeta)\, d\zeta \sum_{m=-\infty}^{\infty} (-1)^m \int_{-\infty}^{\infty} \frac{r^{|m|+1}}{\rho^{|m|+2}} e^{-ik_o\rho} \gamma^*_m(\zeta)\, d\zeta \right\} \tag{91}$$

IV. NUMERICAL COMPUTATION

We consider a family of plasma structures whose properties, in terms of P (r, z), are given by

$$P(r,z) = \frac{P_o(r)}{2} \left[1 + \tanh(\alpha\, k_o z)\right] \tag{92}$$

These configurations have the property of vanishing strength as $z \rightarrow -\infty$ and approaching an axially uniform but radially varying structure

$$P \rightarrow P_o(r)$$

as $z \rightarrow +\infty$. The coefficient α is a parameter which governs the axial gradients or the rate of approach of the plasma properties to their asymptotic values. It is assumed that for an axially uniform plasma with radial variation of P given by $CP_o(r)$ (where C is a constant) the scattered field (the c_m coefficients) is calculable. Then the γ_m which enter into eq. (91) for the back-scattered energy correspond to this set of c_m. The "constant" C is now, however, related to axial location by the factor $\frac{(1 + \tanh \alpha\, k_o z)}{2}$. This gives the strip-theory z-dependence of the γ_m.

Now, as $z \rightarrow -\infty$ the expression (92) for P approaches zero exponentially

$$\underset{z \rightarrow -\infty}{P(r,z) \rightarrow P_o(r)\, e^{2\alpha k_o z} \rightarrow 0} \tag{93}$$

Hence, since the scattering vanishes with P, the γ_m must vanish (also exponentially) as $z \rightarrow -\infty$. This makes the infinite range integrals in the expression (91) for σ_B rapidly convergent in the negative-ζ direction. In the positive-ζ direction, however, the γ_m do not vanish but, rather, settle down to a constant

level as $P \rightarrow P_o(r)$. In order to handle this, the following computational aid is used. The ζ-integrals are broken into two ranges:

$$\int_{\zeta=-\infty}^{\infty} = \int_{\zeta=-\infty}^{z_p} + \int_{\zeta=z_p}^{\infty} \qquad (94)$$

and in the left-hand range, the natural exponential decay of the γ_{nl} simplifies the evaluation of the integral automatically. In the right-hand range, we let

$$T = \zeta - z_p \qquad (95)$$

and the integrals become, generically,

$$\int_{\zeta=z_p}^{\infty} = \int_{T=o}^{\infty} \frac{r^{\nu+1}}{T^2+r^2} e^{\pm ik_o\sqrt{T^2+r^2}} \gamma_n(T+z_p)\, dT \qquad (96)$$

or

$$\int_{\zeta=z_p}^{\infty} = \int_{T=o}^{\infty} \frac{r^{\nu+1}}{(T^2+r^2)^{\frac{\nu+2}{2}}} e^{\pm ik_o\sqrt{T^2+r^2}} \left(\left[\gamma_n(T+z_p)-\gamma_n(\infty)\right] + \gamma_n(\infty)\right) dT \qquad (97)$$

Now, the term in square brackets makes an exponentially decaying contribution to the integral, and the correction term $\gamma_n(\infty)$ can be taken outside the integral, leaving the following expression to be evaluated:

$$I^{\pm} = \gamma_n(\infty) \int_{T=o}^{\infty} \frac{r^{\nu+1}}{(T^2+r^2)^{\frac{\nu+2}{2}}} e^{\pm ik_o\sqrt{T^2+r^2}}\, dT \qquad (98)$$

The value of $\gamma_n(\infty)$ is known from the c_n corresponding to an axially uniform plasma with $P = P_o(r)$. The integral in (98) can be evaluated easily for large radius r. Letting

$$T = r\sinh\theta \qquad (99)$$

$$\left.\begin{aligned} I^{\pm} &= \gamma_n(\infty) \int_{\theta=o}^{\infty} \frac{r^{\nu+1}}{r^{\nu+2}(\cosh\theta)^{\nu+2}} e^{\pm ik_o r\cosh\theta}\, r\cosh\theta\, d\theta \\ &= \gamma_n(\infty) \int_{\theta=o}^{\infty} \frac{e^{\pm ik_o r\cosh\theta}}{(\cosh\theta)^{\nu+1}}\, d\theta \end{aligned}\right\} \qquad (100)$$

For large r, the method of stationary phase[8] gives

$$I^{\pm} \approx \gamma_n(\infty)\sqrt{\frac{\pi}{2k_o r}}\, e^{\pm i\pi/4}\, e^{\pm ik_o r} \qquad (101)$$

Thus the required integrals $\int_{\zeta=-\infty}^{\infty}$ reduce to two exponentially decaying sub-integrals plus expressions of the form $I^{\pm}$ in (101).

As a matter of interest, the availability of the $I^{\pm}$ terms permits computation of the asymptotic values (for large z) of the scattering cross section. The asymptotic value of the integral $\int_{-\infty}^{\infty}(\ )e^{\pm ik_o\sqrt{\ }}d\zeta$ is simply $2I^{\pm}$ for large positive z and zero (0) for large negative z. Moreover, the form of $I^{\pm}$, as shown in Eq. (101), indicates that the positive-z asymptote will be independent of the radius r of the receiver.

A digital computer program has been developed to perform the various phases of the foregoing computation, leading ultimately to σ_B, on the IBM 7090 computer. This program utilizes one subroutine to generate the γ_n functions, which occur in

the integrands of the infinite-range integrals. This subroutine depends upon the type of radial distribution $P_o(r)$ assumed for the plasma electron density. Two options have been programmed for this subroutine, one of which treats a radially uniform plasma, and the other a central uniform core surrounded by a radially nonuniform annulus with electron density varying as $^1/r^2$ and matching up with the core value at the core radius (see Fig. 1c). This corresponds to the configuration (with respect to radius) treated in Ref. (9) except that collisions were not included in the present program. As noted earlier, the γ_n functions are derived from the radial variation but depend on axial location ζ insofar as ζ influences the level or strength of the distribution. The form of the γ_n, as derived in Ref. (9), involves combinations of Bessel functions J_ν, Y_ν, I_ν of both integer and non-integer orders. The necessary Bessel function generation was accomplished by utilizing an existing IBM subroutine. It should be noted that the present program must permit the core plasma to be underdense or overdense, depending upon z, whereas Ref. (9) treated only overdense cores.,

Integrations were performed using numerical quadratures with a continuous test for mesh size. It was found that the integrands exhibited sharp oscillations for small ζ values and then smoothed out as ζ increased in magnitude. In order to avoid unnecessarily lengthy machine runs, a test was introduced into the integration procedure to allow the step size to increase when the integrand smooths out and to cut the step size wherever necessary for accuracy in the region where the integrand is sharply oscillatory. The two series were summed, continuing to increasing values of the summation index, until a stopping criterion

was satisfied. This stopping criterion required three successive partial sums to agree within a preset tolerance. A similar stopping criterion was used to determine the cutoff point for the integrations.

The complete calculations, proceeding from a prescribed configuration and a specified receiver location to a computed value of σ_B, is quite lengthy. This is due to the fact that the integration steps must be quite small in the ζ-region where the integrand oscillation is violent (steps of the order of $\Delta(k_o\zeta) = .005$) whereas the integrations must extend to values of $k_o\zeta$ of the order of 1000. In the regions where the integrand is smooth, step sizes $\Delta(k_o\zeta)$ of the order of 1.5 were permitted. Moreover, for the configuration treated, approximately thirty terms were required in each of the two summations indicated in Eq. (91). As a consequence of this lengthy running time, only a limited number of calculations could be performed.

One configuration was chosen for the sample runs, as described by the following parameter values (see Eq. (92)):

$$P_o(r) = \begin{cases} 2 , & 0 \le k_o r < k_o r_c \\ 2\left(\frac{r_c}{r}\right)^2 , & k_o r_c < k_o r < k_o r_p \\ 0 , & k_o r > k_o r_p \end{cases}$$

$\alpha = .0115$

$k_o r_c = 10$

$k_o r_p = 20$

The P_o value of 2 inside the plasma core, in view of eq. (92) implies that the core changes from underdense to overdense at z= 0. The α-value of .0115 was chosen so that the plasma reaches 99% of its asymptotic value at a z-value corresponding to $k_oz = 200$ (and is reduced to 1% at a z-value corresponding to $k_oz = -200$). The core radius and outer plasma radius values are given by $k_or_c = 10$ and $k_or_p = 20$ respectively. It should be noted that all lengths are dimensionless and are actually scaled with respect to signal wave length, e.g.

$$k_or_p = \frac{2\pi}{\lambda} r_p$$

For this plasma configuration a series of calculations of σ_B (again computed in dimensionless form, $k\sigma_B$) was performed. The results are illustrated by Figs. 3a and 3b and by Table I. At receiver radial distances (from the plasma configuration centerline) corresponding to $k_or = 1500$ and $k_or = 1000$, a survey was made of the back-scattering cross section σ_B as the receiver's axial position z is varied.

As expected, the values of σ_B approach their asymptotes for magnitudes of z, both upstream and downstream, which are comparable to the value, $k_oz=\pm 200$, at which the plasma itself has effectively attained its asymptotic form. For the larger receiver distance ($k_or = 1500$) the scattering has practically achieved its respective asymptotic values for $k_oz = +400$ and for $k_oz = -300$. For the smaller receiver distance (kr=1000) the asymptotes have effectively been attained at $k_oz = +300$ and $k_oz = -200$. The positive z asymptote (computed from the $I^{\pm}$ as noted earlier) has a magnitude $k_o\sigma = 29.47$ in both cases, implying that this is

$$\frac{\cdots}{r^2} \cdots \frac{\cdots}{\partial_r 2} \approx \frac{(-1)\ K_o}{i\pi} \int_{-\infty} \frac{r^{\cdots}}{\rho^{n+3}}\ e^{iK_o\rho}\ e^{-in\pi/2}\ g_{-n}(\zeta)\, d\zeta$$

the back-scattering cross section for the asymptotic configuration given by

$$P(r) = \begin{cases} 2, & 0 < k_or < 10 \\ 2\left(\frac{r_c}{r}\right)^2 & 10 < k_or < 20 \\ 0 & k_or > 20 \end{cases}$$

The wavelike irregularity with respect to axial location, in the back scattering at the closer receiver is quite noticeable, whereas the more distant receiver exhibits a smoother behavior as it traverses an axial path relative to the plasma configuration. This smoothing trend should continue in the direction of increasing receiver radius and, while a more severe wave may build up as receiver distance is decreased, the asymptotic approximation used in the calculation of the spherical Hankel functions does not permit a closer approach without an attendant loss of accuracy. The waviness in the axial dependence of the return is undoubtedly caused by a spatially oscillatory diffraction pattern superimposed on the monotonic behavior $\frac{1}{2}(1+\tanh \alpha k_oz)$ characterizing the plasma itself. This diffraction pattern could be likened to a lobe pattern for the scattered field, the more distant receiver sensing this pattern in a more diffuse form with respect to linear distance, and the closer receiver sensing it in a more concentrated form spatially.

Figs. 2a and 2b illustrate (by means of plots of successive partial sums) the mode of convergence of the series for the axial electric field E_z (for both real and imaginary components) for both receiver distances, $k_or = 1000$ (Fig. 2a) and $k_or = 1500$ (Fig. 2b). The mode of convergence is seen to be quite erratic

in both cases until the summation index exceeds a value of about 20, following which, the partial sums settle down rapidly. In this connection one remark is of interest: some attempts have been made by other organizations recently to calculate approximately the scattering from overdense turbulent plasmas using a sort of Born scattering but assuming the electrons excited in accordance with the deterministic field corresponding to the mean plasma configuration. In *these calculations only the zero-order* ($m = 0$ *in the summation*) *term was used.* Fig. 2 illustrates the magnitude of the error that can be introduced by stopping at any index value short of the converged values.

The results presented apply for the specific sample configuration treated. The computer program which has been developed for this calculation, however, has the capability to handle other configurations and, with appropriate substitution of the γ_m-subroutine, other radial distributions of electron density. As noted earlier, a γ_m-subroutine is available for treating radially uniform distributions. Other, more general radial distributions might require numerical solution of the radial differential equation for each (θ) Fourier component, but this involves no conceptual difficulties. The radial solution would be performed by finite-difference technique and would replace the Bessel function generation appropriate to the γ_m-calculation for the present electron density radial distribution.

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LIST OF SYMBOLS

c	light speed = $1/\sqrt{\epsilon_o \mu_o}$
e	electron charge
$\vec{E}$	electric field
E_o	incident electric field intensity
$\vec{H}$	magnetic field
k_o	$= \omega/c$
m	electron mass
n	electron number density
$P(r,z)$	$= \frac{\omega_p^2}{\omega^2+\nu^2}\left(1-\frac{i\nu}{\omega}\right)$
r, θ, z	cylindrical coordinates
R, θ, φ	spherical polar coordinates
x, y, z	cartesian coordinates
r_c	plasma core radius
r_p	external plasma radius
$\vec{S}$	Poynting vector
ϵ_o	dielectric permeability of vacuum
μ_o	magnetic permeability of vacuum
ω	signal frequency
ω_p	plasma frequency = $\sqrt{\frac{e^2 n}{\epsilon_o m}}$
ν	collision frequency (electrons with neutral particles)

σ_B	back scattering cross section
λ	signal wave length = $2\pi/k_o$
φ, χ	defined by Eqs. (16), (17), (18)
ρ	distance between field and source points
$(\ )_{x,y,z}$	cartesian components of a vector
$(\ )_{r,\theta,z}$	cylindrical components of a vector

TABLE I

Numerical Values of Back-Scattering Cross Sections

$k_o r = 1000$

$k_o z$	$k_o \sigma_B$
- 00	0
-200	.3398
-150	1.9186
-100	7.1827
- 75	9.3858
- 62.5	10.8322
- 50	11.155
- 37.5	9.9628
- 25	8.1983
- 12.5	7.2458
0	7.9930
25	13.736
50	20.449
100	26.282
150	27.490
200	28.516
00	29.47

$k_o r = 1500$

$k_o z$	$k_o \sigma_B$
- 00	0
-300	.3325
-200	2.380
-140	6.477
-100	7.6145
- 50	7.300
0	8.683
25	14.148
50	19.521
100	24.5467
200	27.8836
300	29.270
00	29.47

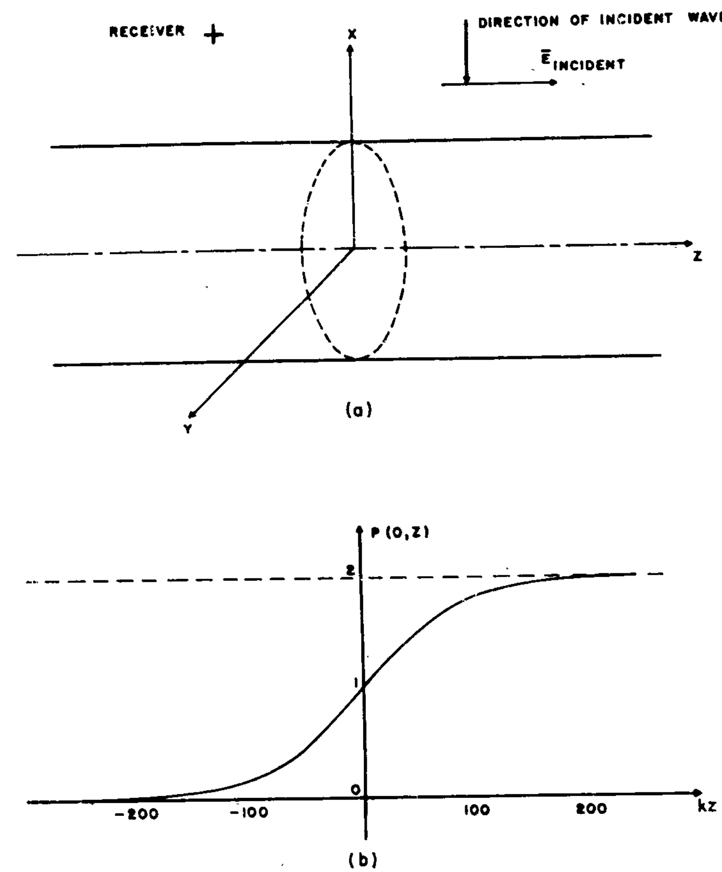

FIG. 1 AXIAL VARIATION OF CENTERLINE VALUE OF ELECTRON DENSITY

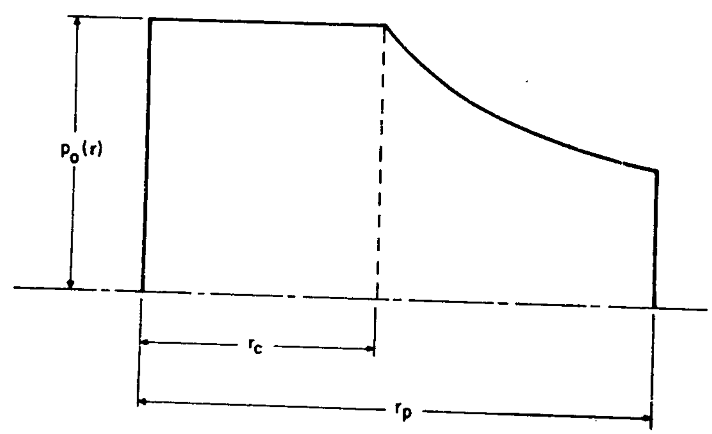

FIG. Ic PLASMA CONFIGURATION – RADIAL VARIATION

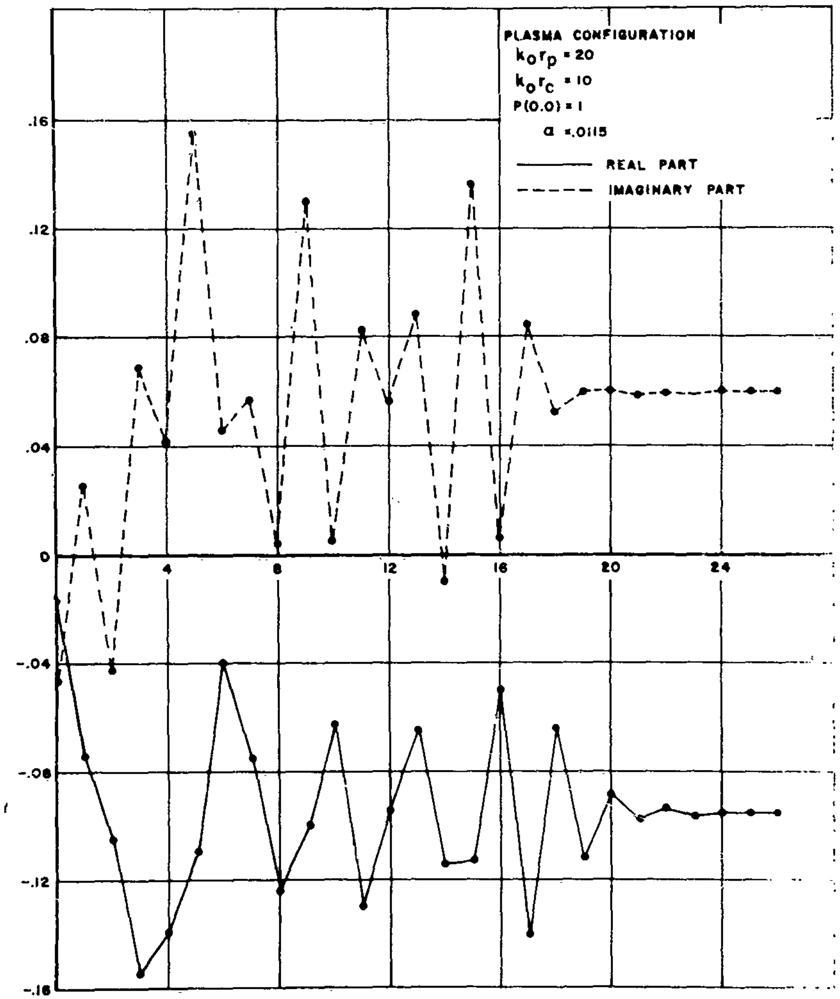

FIG. 2a PARTIAL SUMS VERSUS NUMBER OF TERMS IN SERIES FOR E_Z AT $k_o r = 1000$ $k_o z = 0$

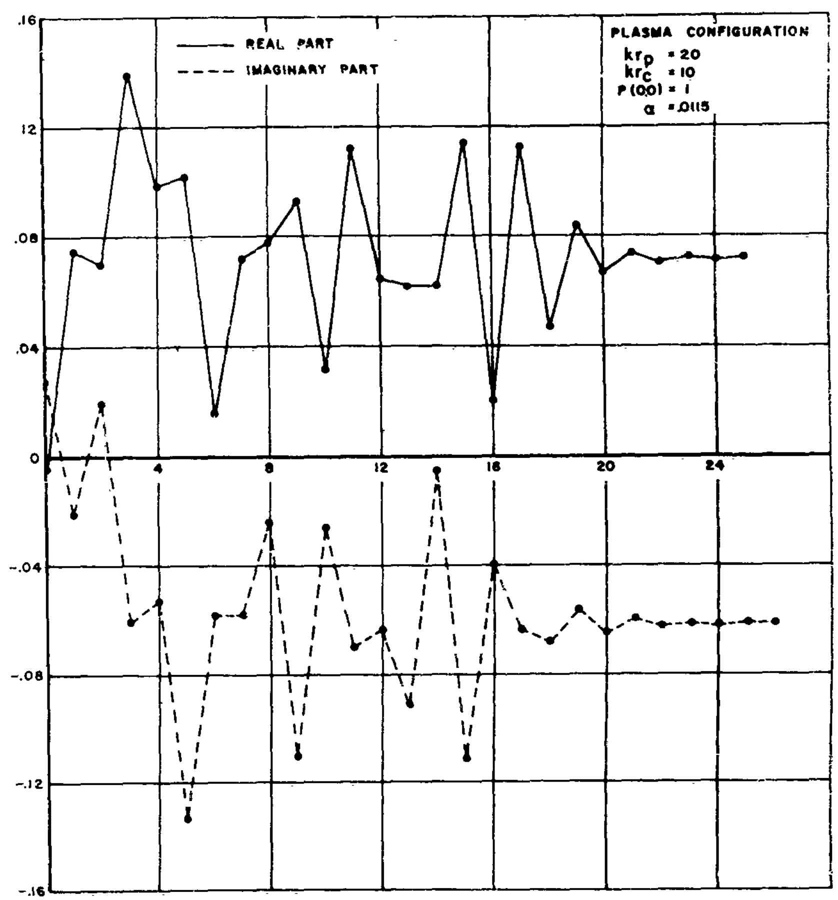

FIG. 2-b PARTIAL SUMS VERSUS NUMBER OF TERMS IN SERIES FOR E_Z AT $k_o r = 1500$ $k_o z = 0$

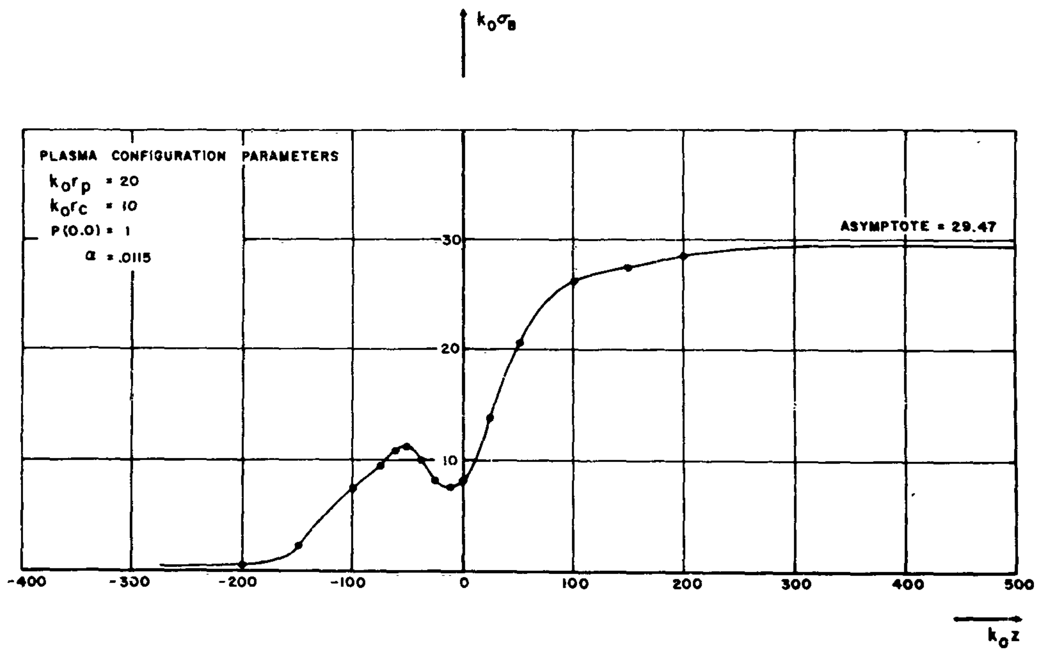

FIG. 3a BACK SCATTERING CROSS SECTION VERSUS AXIAL POSITION OF RECEIVER FOR DIMENSIONLESS RADIAL DISTANCE $k_0r = 1000$

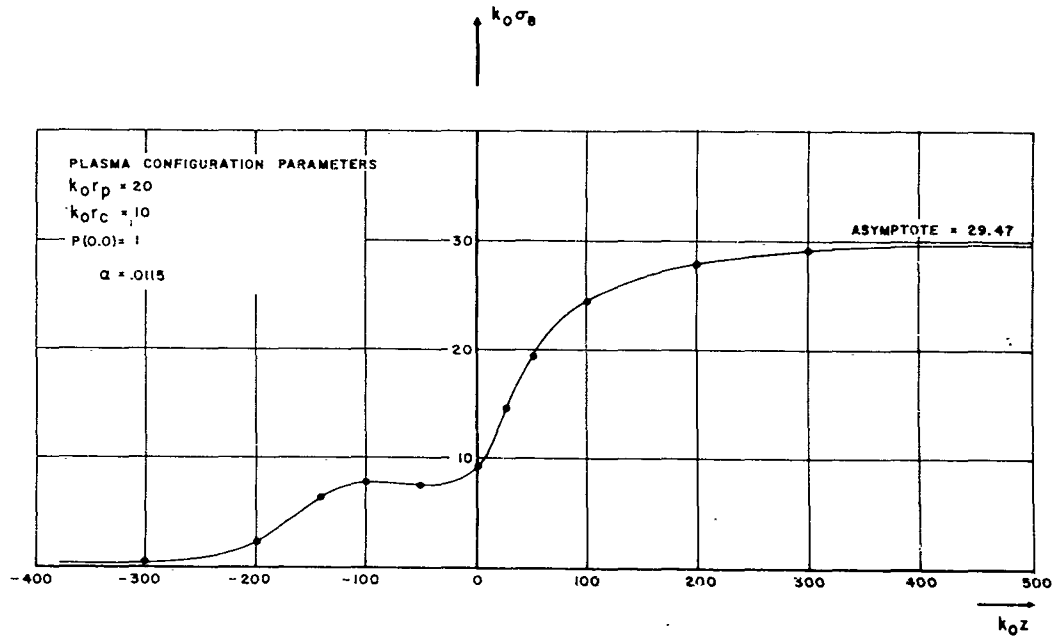

FIG. 3b BACK SCATTERING CROSS SECTION VERSUS AXIAL POSITION OF RECEIVER FOR DIMENSIONLESS RADIAL DISTANCE $k_0 r = 1500$

END